JOURNAL OF NUMBER THEORY 40, 329–335 (1992)

A Conditional Resolution of the Parity Problem in Sieve Theory

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Communicated by Hans Zassenhaus

Received May 24, 1989; revised July 3, 1990

DEDICATED TO PROFESSOR PAUL BATEMAN

1. INTRODUCTION

Let N be a large positive even number and p, p_1, p_2 be primes. We are concerned with the parity problem for the sequences $\{N-p; p<N\}$ and $\{p+2\}$. The parity problem refers to the ambiguity between integers with an odd number of prime factors and those with an even number. The exceptional difficulty of this problem has already been observed by Selberg [9] and Bombieri [1]. The work of Bombieri shows that even assuming the Elliott–Halberstam conjecture, it is impossible for sieve methods to resolve the parity problem for the sequence $\{p+2\}$. In this paper, we shall give a conditional resolution of the parity problem.

Let $\phi(d)$ be the Euler function,

$$\pi(y; d, l) = \sum_{\substack{p \leqslant y \\ p \equiv l \pmod d}} 1$$

and

$$li\, y = \int_2^y \frac{dt}{\log t}.$$

Inequalities of the type

$$\sum_{d \leqslant x^\theta} \max_{y \leqslant x} \max_{(l,d)=1} \left| \pi(y; d, l) - \frac{li\, y}{\phi(d)} \right| \ll \frac{x}{(\log x)^A} \qquad (A>0) \tag{1}$$

have played an important role in sieve theory. The well-known theorem of Bombieri and Vinogradov asserts that (1) is admissible with $\theta = 1/2 - \varepsilon$.

0022-314X/92 $3.00

Elliott and Halberstam have conjectured that (1) is admissible even with $\theta = 1 - \varepsilon$. Recently, the strong form of the Elliott and Halberstam conjecture has been disproved by Friedlander and Granville [5]. However, they believe that (1) does hold for $\theta = 1 - \varepsilon$. We shall assume this conjecture. In addition, we shall assume the validity of the estimate

$$|\{p+2; p \leqslant x, p+2 = p_1 p_2 p_3, p_i \geqslant x^\alpha, i = 1, 2, 3\}| \leqslant \beta I C x \log^{-2} x, \quad (2)$$

where α is any number in the interval $(1/4, 1/3)$, β is a number very slightly less than $2/3$,

$$C = 2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right)$$

and

$$I = \int_\alpha^{1-2\alpha} \frac{\log((1-t)/\alpha - 1)}{t(1-t)}\, dt.$$

Assuming the conjecture of Elliott and Halberstam, it can be shown that the expression in (2) is

$$\leqslant (\tfrac{2}{3} + \varepsilon)\, I C x \log^{-2} x.$$

We shall prove the following

THEOREM. *Assume* (1) *holds with* $\theta = 1 - \varepsilon$ *and* (2) *is valid. Then*

$$|\{p+2; p \leqslant x, p+2 = p_1 p_2\}| \geqslant (2 - 3\beta - \varepsilon_1)\, I C x \log^{-2} x,$$

where ε_1 *goes to zero as* x *approaches infinity.*

So under the above hypotheses, we can show that there are infinitely many primes p such that $p+2$ has exactly two prime factors. Furthermore, it will be seen from the proof that one of the prime factors lies in the interval $[x^\alpha, x^{1-2\alpha}]$. Of course, our Theorem can be formulated for the sequence $\{N - p; p < N\}$. Obviously, these results are closely related to the twin-prime and the Goldbach conjectures.

2. SEVERAL LEMMAS

Let $\mathscr{A}$ be a finite sequence of integers, $\mathscr{P}$ be a set of primes, and $|\mathscr{A}|$ the number of elements in $\mathscr{A}$.

In addition, for a positive integer d, suppose that the quantity

$$|\mathscr{A}_d| = |\{a \in \mathscr{A}; a \equiv 0 \pmod{d}\}|$$

may be written in the form

$$|\mathscr{A}_d| = \frac{\omega(d)}{d} X + r_d \qquad (\mu(d) \neq 0),$$

where $\omega(d)$ is multiplicative with $0 \leqslant \omega(p) < p$, X is a large enough parameter independent of d, r_d is an error term, and $\mu(d)$ is the Möbius function.

Finally, for a given $z \geq 2$, we let

$$P(z) = \prod_{\substack{p < z \\ p \in \mathscr{P}}} p,$$

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{\substack{a \in \mathscr{A} \\ (a, P(z)) = 1}} 1,$$

and

$$W(z) = \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right).$$

We now state two lemmas.

LEMMA 1. (Halberstam–Jurkat–Richert). *Suppose $\omega(d)$ satisfies the conditions*

$$0 \leqslant \frac{\omega(p)}{p} < 1 - \frac{1}{A_1},$$

$$-A_2 \leqslant \sum_{w \leqslant p \leqslant z} \frac{\omega(p)}{p} \log p - \log \frac{z}{w} \leqslant A_3, \qquad 2 \leqslant w \leqslant z,$$

for some suitable constants $A_i \geq 1$, $i = 1, 2, 3$. For $\xi \geqslant z$ we have

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XW(z) \left\{F\left(\frac{\log \xi^2}{\log z}\right) + O\left(\frac{A_2}{(\log \xi)^{1/14}}\right)\right\} + R, \tag{3}$$

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant XW(z) \left\{f\left(\frac{\log \xi^2}{\log z}\right) + O\left(\frac{A_2}{(\log \xi)^{1/14}}\right)\right\} - R, \tag{4}$$

where $R = \sum_{d < \xi^2, d \mid P(z)} 3^{\nu(d)} |r_d|$, $\nu(d)$ is the number of distinct prime factors of d, the functions F and f are defined by

$$F(u) = \frac{2e^{\gamma}}{u}, \qquad f(u) = 0 \quad \textit{for} \quad 0 < u \leqslant 2$$

and

$$(uF(u))' = f(u-1), \qquad (uf(u))' = F(u-1)$$

for $u \geq 2$. *Here* γ *is Euler's constant.*

This lemma is also true if

$$1 < \xi < z \qquad \text{but} \qquad z \ll \xi^{\lambda}$$

with a positive constant λ, in which case the O-constant in (3) depends also on λ.

The proof of this lemma can be found in [6, Chap. 8].

It is well-known that

$$F(u) = \frac{2e^{\gamma}}{u}, \qquad 0 < u \leqslant 3, \tag{5}$$

and

$$f(u) = \frac{2e^{\gamma}}{u} \log(u-1), \qquad 2 \leqslant u \leqslant 4. \tag{6}$$

The next lemma is a consequence of the conjecture of Elliott and Halberstam.

LEMMA 2. *Assume* (1) *holds with* $\theta = 1 - \varepsilon$. *Then, given any positive constant* A, *there exists* $\varepsilon > 0$ *such that*

$$\sum_{d \leqslant x^{1-\varepsilon}} \mu^2(d)\, 3^{\nu(d)} \max_{y \leqslant x} \max_{(l, d) = 1} \left| \pi(y; d, l) - \frac{li\, y}{\phi(d)} \right| \ll \frac{x}{(\log x)^A}.$$

3. PROOF OF THEOREM

We take $\mathscr{A} = \{p + 2; p \leqslant x\}$, $\mathscr{P} = \{p; p > 2\}$, $\omega(p) = p/(p-1)$ for $p \in \mathscr{P}$. By Mertens' formula and the definition of C,

$$W(z) \sim e^{-\gamma} C (\log z)^{-1} \qquad \text{as} \quad z \to \infty.$$

Throughout this section we assume the validity of the conjecture of Elliott and Halberstam and the estimate (2).

Recall $\alpha \in (1/4, 1/3)$. We now fix the parameters.

$$z = x^{\alpha}, \quad \alpha_1 = 1 - 2\alpha, \quad z_1 = x^{\alpha_1}, \quad I = \int_{\alpha}^{\alpha_1} \frac{\log((1-t)/\alpha - 1)}{t(1-t)}\, dt.$$

LEMMA 3. $|\{p+2;\ p \leqslant x,\ p+2=p_1 \text{ or } p_1p_2,\ p_i \geqslant z,\ i=1,2\}| \geqslant S(\mathscr{A}, \mathscr{P}, z) - \beta ICx \log^{-2} x$.

Proof. This follows from the definition of $S(\mathscr{A}, \mathscr{P}, z)$ and the estimate (2).

LEMMA 4. $S(\mathscr{A}, \mathscr{P}, z_1) \leqslant S(\mathscr{A}, \mathscr{P}, z) - (2-2\beta-\varepsilon_1)\, ICx \log^{-2} x$.

Proof. Buchstab's identity, when applied twice, gives the equality

$$S(\mathscr{A}, \mathscr{P}, z_1) = S(\mathscr{A}, \mathscr{P}, z) - \sum_{z \leqslant p_1 < z_1} S(\mathscr{A}_{p_1}, \mathscr{P}, z) + \sum_{z \leqslant p_2 < p_1 < z_1} S(\mathscr{A}_{p_1p_2}, \mathscr{P}, p_2).$$

To estimate

$$\sum_{z \leqslant p_1 < z_1} S(\mathscr{A}_{p_1}, \mathscr{P}, z)$$

from below, we apply (4) of Lemma 1 with

$$X = \frac{\omega(p_1)}{p_1}\, li\, x, \qquad z = x^{\alpha}, \qquad \xi^2 = \frac{x^{1-\varepsilon}}{p_1}.$$

Using Lemma 2, the Prime Number Theorem, and the definition of $f(u)$ for $2 \leq u \leq 4$, we find

$$\sum_{z \leqslant p_1 < z_1} S(\mathscr{A}_{p_1}, \mathscr{P}, z) \geqslant (2-\varepsilon_1)\, ICx \log^{-2} x.$$

Next we turn to

$$\sum_{z \leqslant p_2 < p_1 < z_1} S(\mathscr{A}_{p_1p_2}, \mathscr{P}, p_2).$$

A moment's thought shows that this sum is equal to

$$2|\{p+2; p \leqslant x, p+2 = p_1p_2p_3, p_i \geqslant z, i=1,2,3\}| + O(x^{2\alpha_1}).$$

Lemma 4 now follows when we appeal to the estimate (2) and combine the above estimates.

We now proceed to prove the Theorem of the paper. Comparing Lemmas 3, 4 and using the fact

$$\beta < 2 - 2\beta - \varepsilon_1 \qquad \text{as} \quad x \to \infty,$$

we deduce that

$$|\{p+2; p \leqslant x, p+2=p_1 p_2\}| \geqslant (2-3\beta-\varepsilon_1)\, ICx \log^{-2} x.$$

Here ε_1 goes to zero as x approaches infinity. This completes the proof of the Theorem.

4. An Upper Bound

In this section, assuming the weak Elliott–Halberstam conjecture, we derive an upper bound for the expression in (2). We have

$$\begin{aligned}&|\{p+2; p \leq x, p+2=p_1 p_2 p_3, p_i \geq z, i=1,2,3\}|\\ &\leq \frac{1}{6} \sum_{\substack{z \leq p_1 \leq z_1\\ p_1 \neq p_2}} \sum_{z \leq p_2 \leq x^{1-\alpha}/p_1} S(\mathscr{A}_{p_1 p_2}, \mathscr{P}, z) + O(x^{1-\alpha}). \qquad (7)\end{aligned}$$

We now apply (3) of Lemma 1 with

$$X = \frac{\omega(p_1 p_2)}{p_1 p_2}\, li\, x, \qquad z = x^{\alpha}, \qquad \xi^2 = \frac{x^{1-\varepsilon}}{p_1 p_2}.$$

Using Lemma 2, partial summation, and the definition of $F(u)$ for $1 \leq u \leq 3$, we see that (7) is

$$\begin{aligned}&\leq \frac{1}{6}(2+\varepsilon_1)\, Cx \log^{-2} x \cdot \int_{\alpha}^{1-2\alpha} \int_{\alpha}^{1-\alpha-t_1} \frac{dt_2\, dt_1}{t_1 t_2 (1-t_1-t_2)}\\ &\leq \left(\frac{2}{3}+\varepsilon_1\right) Cx \log^{-2} x \cdot \int_{\alpha}^{1-2\alpha} \frac{\log((1-t_1)/\alpha - 1)}{t_1(1-t_1)}\, dt_1,\end{aligned}$$

as claimed in the Introduction.

Acknowledgments

I am grateful to Professors R. J. Miech and H.-E. Richert for their encouragement, and to Professors H. Halberstam and H.-E. Richert for several discussions. I also thank the Department of Mathematical Sciences at the University of Texas at El Paso for their support.

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